

THE MASCHKE PROPERTY FOR THE SYLOW p -SUBGROUPS OF THE SYMMETRIC GROUP S_{p^n}

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ABSTRACT. The Sylow p -subgroups of the symmetric group S_{p^n} satisfy the appropriate generalization of Maschke's Theorem to the case of a p' -group acting on a (not necessarily abelian) p -group. Moreover, some known results about the Sylow p -subgroups of S_{p^n} are stated in a form that is true for all primes p .

1. INTRODUCTION

Several authors have considered generalizations of Maschke's Theorem in the context of groups acting on groups. For example, the case of coprime action on an elementary abelian p -group is found in [12]; and in [1] Berkovich studied the case of abelian V , see also [2, §6]. Here we study a more general Maschke property for finite groups.

Definition 1.1 (Maschke property). A π -group V has the Maschke property if for every π' -group G acting on V the following property holds: if N is a G -invariant normal subgroup of V which has a complement in V , then it has a G -invariant complement.

We call it a property rather than a theorem because it does not hold for all groups: one counterexample is an action of the cyclic group C_3 on the 2-group $Q_8 * C_4$, a central product (Example 2.1). But all abelian groups are Maschke [2, §6], as are all metacyclic p -groups (Proposition 2.4). Our main result is:

Theorem 1.2. *For every prime p the Sylow p -subgroups of the symmetric group S_{p^n} have the Maschke property.*

Proving Theorem 1.2 requires some properties of these Sylow p -subgroups which are presumably well known, but may never have been written down in an easily accessible form for all primes p . The first such result is:

Proposition 1.3 (various authors). *Let P_n be a Sylow p -subgroup of S_{p^n} for any prime p . Then:*

- (1) $C_{S_{p^n}}(P_n) = Z(P_n)$ and $N_{S_{p^n}}(P_n)/P_n \cong C_{p-1}^n$.
- (2) $\text{Aut}(P_n)$ has a normal Sylow p -subgroup, with factor group C_{p-1}^n .

So for $p = 2$, $\text{Aut}(P_n)$ is a 2-group and P_n is self-normalizing in S_{p^n} .

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Remark 1.4. Part 1) for odd primes was proved by Cárdenas and Lluís [4]. Both [7] and [2, Corollary A.13.3] say that P. Hall proved the case $p = 2$ in 1956. A modern treatment may be found in [2, Appendix 13].

Turning to 2), Bodnarchuk described the full structure of $\text{Aut}(P_n)$ for odd primes [3], whereas we have not yet located a proof for $p = 2$.

That reduces the proof of Theorem 1.2 to the case of coprime action in S_{p^n} , where we will prove the following result:

Theorem 1.5. *Let P_n be a Sylow p -subgroup of S_{p^n} for any prime p . Then there is a Hall p' -subgroup H of $N_{S_{p^n}}(P_n)$ which has the following property:*

If $N \trianglelefteq P_n$ is a normal subgroup which has a complement in P_n , then N has an H -invariant complement in P_n .

As N is not required to be H -invariant, this is a strengthening of the Maschke property. For $p^n = 3^3$, Example 11.4 constructs a complemented normal subgroup which does indeed fail to be H -invariant. This strengthening is false in the original context of Maschke's Theorem:

Example 1.6. The dihedral group D_8 has an irreducible ordinary representation in degree two. Every one-dimensional subspace of the representation has a complement, but by irreducibility there is no invariant complement.

We can now proceed to the proof of Theorem 1.2. One step of the proof will be used again later, so we turn it into a remark.

Remark 1.7. Suppose that the finite group G has a normal Sylow p -subgroup Q , with G/Q is abelian. Observe that G is solvable, so by a theorem of P. Hall [8, Thm 6.4.1, p. 231], G has a Hall p' -subgroup H , and every p' -subgroup of G is conjugate to a subgroup of H . Observe that H is isomorphic to G/Q .

Proof of Theorem 1.2. Since $C_{S_{p^n}}(P_n) = Z(P_n)$, the Hall p' -subgroup H of Theorem 1.5 embeds in $\text{Aut}(P_n)$. Proposition 1.3 says that H is also a Hall p' -subgroup of $\text{Aut}(P_n)$, and so $\text{Aut}(P_n)$ is solvable. By Remark 1.7, every p' -subgroup of $\text{Aut}(P_n)$ is conjugate to a subgroup of H . The result follows by Theorem 1.5. \square

We include a proof of Proposition 1.3 for two reasons: the $p = 2$ case of part 2) may not be in the literature; and the proof of Theorem 1.5 necessitates our constructing explicit generators for a Hall p' -subgroup of $N_{S_{p^n}}(P_n)$.

Recall that P_n is the n -fold iterated wreath product $C_p \wr C_p \wr \cdots \wr C_p$, and so $P_n \cong C_p \wr P_{n-1}$. In our proof of Proposition 1.3 2) we will use the following result, which is due to Weir for $p \neq 2$:

Proposition 1.8. *Let p be an arbitrary prime. Then P_n has a characteristic abelian subgroup B with the following properties:*

- (1) $P_n/B \cong P_{n-1}$
- (2) *The action of $P_{n-1} \cong P_n/B$ on B is uniserial.*

Remark 1.9. Weir [15] proved this for $p \neq 2$; for B he used the subgroup which he calls A^{n-1} (see Notation 3.5 below), and which Huppert constructs in [10, III.15.4 Satz a), p. 380]. For $p = 2$, Huppert constructs our B in [10, III.15.4 Satz b), p. 381]. He only remarks that it is abelian, normal and not contained in A^{n-1} ; Covello shows that it is characteristic [5, Thm 4.4.6]. For $p^n = 2^3$, our B is the group \mathfrak{H}_7 which Dmitruk constructs in [6, p. 124].

The proofs of Theorem 1.5 and Proposition 1.8 make use of the following result. See Section 4 below for Weir's terminology T_j and "depth".

Proposition 1.10. *Let p be a prime. If $N \trianglelefteq P_n$ has depth j , then $[T_j, T_j] \leq N$.*

Remark 1.11. Weir [15, Thm 4] proved this under the assumption that p is odd and N a partition subgroup. See also Dmitruk's [6, Thm 5a)] for the case where $p = 2$ and N is characteristic.

Structure of the paper. In Section 2 we give some first examples and counterexamples for the Maschke property. Section 3 recalls the identification of P_n as an iterated wreath product, introduces the generators σ_i and recalls Weir's subgroup A^{n-1} . Next we recall Weir's filtration T_j in Section 4 and prove Proposition 1.10, followed in the next section by the proof of Proposition 1.8. After this, we construct the Hall subgroup for Theorem 1.5 in Lemma 6.1 and prove Proposition 1.3.

The proof of Theorem 1.5 occupies the next four sections. If $N \trianglelefteq P_n$ has depth j , then it contains $K := [T_j, T_j]$ by Proposition 1.10, and N/K is an $\mathbb{F}_p P_j$ -submodule of T_j/K . Now, T_j/K is a direct sum of $n-j$ copies of the uniserial module A^j ; and if N has a complement in P_n , then N/K is a direct summand of T_j/K . So in Section 7 we suppose that M is any uniserial module; we characterise which submodules of M^n have complements, and show that if N has a complement then it has one of the form M_Z . In Sections 8 and 9 we apply this general theory in the case $M^n = T_j/K$. In particular we establish a necessary condition on N (Lemma 9.3, which builds on Lemmas 8.3 and 9.2), without which N cannot have a complement, even if N/K does. Finally in Section 10 we construct certain permutations ρ_i and use them to show that if N/K has a complement and N satisfies the necessary condition of Lemma 9.3, then the complement M_Z of N/K lifts to an H -invariant complement of N , concluding the proof of Theorem 1.5.

The paper ends with an extensive selection of examples, and the application of our results to Weir's partition subgroups. In an appendix we briefly consider the largest abelian subgroups of P_n .

2. THE MASCHKE PROPERTY: FIRST EXAMPLES

First we give counterexamples of p -rank two for $p = 2, 3$. The counterexample for $p = 3$ is also of maximal class.

Example 2.1. For $p = 2$ let V be the central product $V = Q_8 * C_4$. That is, $V = \langle i, j, k, x \rangle$ with x central, $x^2 = -1$ and $|V| = 16$. Observe that V has 2-rank two.

There is an automorphism ϕ of order 3 which acts on the set $\{i, j, k, x\}$ as the 3-cycle $(i j k)$. So $G = \langle \phi \rangle \cong C_3$ acts coprimely on V , and $N = Q_8 = \langle i, j, k \rangle$ is a G -invariant normal subgroup which has a complement: each of the six involutions $\pm ix, \pm jx, \pm kx$ generates a complement. But ϕ acts on this set of six complements as a permutation of type 3^2 , and there are no other complements. So $Q_8 * C_4$ does not have the Maschke property.

Example 2.2. Let V be the semidirect product $V = (\mathbb{Z}/9\mathbb{Z})^2 \rtimes C_3$, where the action of $C_3 = \langle x \rangle$ on $(\mathbb{Z}/9\mathbb{Z})^2$ is as follows:

$$xv = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{for} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Observe that V has order 3^5 ; it has 3-rank two; and it is of maximal class.

Now, $V' = \{v \in (\mathbb{Z}/9\mathbb{Z})^2 \mid v_1 \in 3\mathbb{Z}/9\mathbb{Z}\}$, which has order 3^3 . Consequently, $N := \langle V', x \rangle$ is a normal subgroup of order 3^4 . Moreover,

$$v + {}^xv + {}^{x^2}v = 0 \quad \text{for every } v \in (\mathbb{Z}/9\mathbb{Z})^2,$$

and so (v, x) has order 3 for every $v \in (\mathbb{Z}/9\mathbb{Z})^2$. So $C_v := \langle (v, x) \rangle$ is cyclic of order 3 for every $v \in (\mathbb{Z}/9\mathbb{Z})^2$; and C_v is a complement of N in V for every $v \in (\mathbb{Z}/9\mathbb{Z})^2 \setminus V'$. As every $v \in (\mathbb{Z}/9\mathbb{Z})^2 \setminus V'$ has order 9, it follows that every complement of N in V is a C_v with $v \in (\mathbb{Z}/9\mathbb{Z})^2 \setminus V'$.

By construction of V , $\alpha(v, x) := (-v, x)$ defines an automorphism of V , of order 2. Then N is α -invariant, and $\alpha(C_v) = C_{-v}$. As $C_v \neq C_{-v}$ for $0 \neq v \in (\mathbb{Z}/9\mathbb{Z})^2$, it follows that N has no $\langle \alpha \rangle$ -invariant complement in V . So V does not have the Maschke property.

The following lemma will be used to prove the metacyclic and rank two cases of Proposition 2.4 below.

Lemma 2.3. *If G acts coprimely on the regular p -group V , and if $N \trianglelefteq V$ is a G -invariant normal subgroup which has a cyclic complement in V , then N has a G -invariant complement in V .*

Proof. Let L be a cyclic complement of N in V , and let $|L| = p^\ell$. Set $V_1 := \Omega_\ell(V) = \langle g \in V \mid g^{p^\ell} = 1 \rangle$, which is characteristic and hence G -invariant. Then $L \leq V_1$, and L is a complement in V_1 to $N_1 := N \cap V_1$. Any G -invariant complement to N_1 in V_1 will be a complement to N in V too.

As N_1 has a cyclic complement, [9, Prop 5.2] says that there is a G -invariant cyclic subgroup $C \leq V_1$ with $N_1 C = V_1$. We want $N_1 \cap C = 1$. Now, $|C : C \cap N_1| = |V_1 : N_1| = |L| = p^\ell$, so if $N_1 \cap C \neq 1$ then the cyclic group $C \leq V_1$ has order $> p^\ell$. But as V is regular and $V_1 = \Omega_\ell(V)$, [10, 10.5 Hauptsatz p. 324] says that $V_1 = \{g \in V \mid g^{p^\ell} = 1\}$. So $C \cap N_1 = 1$ and C is the desired G -invariant complement. \square

Proposition 2.4. *Let V be a finite group. If*

- (1) *V is abelian; or*
- (2) *V is a metacyclic p -group; or*
- (3) *V is a p -group of p -rank two, for $p > 3$*

then V has the Maschke property.

Remark 2.5. Bettina Wilkens has shown us an argument demonstrating that every metacyclic finite group has the Maschke property.

Proof. Suppose that G acts coprimely on V , and that the G -invariant normal subgroup $N \trianglelefteq V$ has complement L in V . Assume $N \neq 1$.

V abelian: This is known, but we give the proof for the sake of completeness. Consider V as a $\mathbb{Z}_{(p)}$ -module. As $V = N \times L$, there is a $\mathbb{Z}_{(p)}$ -linear $\pi : V \rightarrow N$ with $\pi|_N = \text{Id}$. Since $|G|$ is a unit in $\mathbb{Z}_{(p)}$, the usual proof of Maschke's Theorem means that π can be chosen to be $\mathbb{Z}_{(p)}G$ -linear.

V metacyclic, $p = 2$: By [14, Lemma 1], if G acts nontrivially and V is nonabelian then $V = Q_8$. But then only $N = 1$ and $N = V$ have complements.

V metacyclic, p odd: If $[V, N] = 1$ then $V' = L'$. Hence V/L' is abelian, and there is $L' \leq W \trianglelefteq V$ with W/L' a G -invariant complement to $NL'/L' \cong N$. So W is a G -invariant complement to N .

So we assume $[V, N] \neq 1$. Let $K \leq V$ be cyclic with V/K cyclic, so $V' \leq K$ and V is a regular p -group by [10, III.10.2 Satz p. 322]. Since N is normal and $V' \leq K$ we have $K \cap N \neq 1$. As K is cyclic and $N \cap L = 1$, it follows that $L \cap K = 1$. Therefore $L \cong LK/K \leq V/K$ is cyclic, and the result follows by Lemma 2.3. V has p -rank two, $p \geq 5$: Set $F = N \cap \Omega_1(Z(V))$; from $N \neq 1$ it follows that $F \neq 1$. If $E \leq L$ is elementary abelian then EF is elementary abelian too. As $E \cap N = 1$ it follows that EF has rank larger than that of E . It follows that L has p -rank one. So L is cyclic, by [10, III.8.2 Satz p. 310].

By Lemma 2.3 it suffices to show that V is regular. By a theorem of Blackburn [10, III.12.4 Satz p. 343], V satisfies one of three conditions. In Blackburn's Case (1), V is metacyclic. In Case (2), $V' = \langle Z^{p^{n-3}} \rangle$ is cyclic; and in Case (3), V has nilpotency class $3 < p$. So V is regular in cases (2) and (3) by a) and c) of [10, III.10.2 Satz p. 322]. \square

3. THE ITERATED WREATH PRODUCT

Recall that if $S \leq \text{Sym}(X)$ and $G \leq \text{Sym}(Y)$ are permutation groups acting on finite sets, then there is a wreath product group

$$G \wr S = G^{|X|} \rtimes S \leq \text{Sym}(Y \times X)$$

with S -action given by $({}^\sigma \underline{g})_x = g_{\sigma^{-1}(x)}$ for $\sigma \in S$, $\underline{g} \in G^{|X|}$ and $x \in X$. By [10, I.15.4 Hilfssatz, p. 96] we have associativity: $G \wr (S \wr T) \cong (G \wr S) \wr T$.

Let p be a prime number. The cyclic group C_p embeds in the symmetric group S_p as the subgroup generated by a p -cycle, and so the n -fold iterated wreath product

$$P_n := \underbrace{C_p \wr C_p \wr \cdots \wr C_p}_{n \text{ copies of } C_p}$$

embeds in S_{p^n} . Kaloujnine (in [11]; see also [10, III.15.3 Satz, p. 378]) proved that P_n is a Sylow p -subgroup of S_{p^n} .

We shall treat S_n as the group of permutations of $\{0, 1, \dots, n-1\}$ rather than of $\{1, 2, \dots, n\}$. Using the p -adic representation $a = \sum_{i=0}^{n-1} b_i p^{n-1-i}$ we can identify $a \in \{0, 1, \dots, p^n - 1\}$ with $(b_0, b_1, \dots, b_{n-1}) \in \mathbb{F}_p^n$. In particular, we may identify S_{p^n} with the symmetric group $\text{Sym}(\mathbb{F}_p^n)$.

Lemma 3.1. *Denote by σ the p -cycle $\sigma = (0 \ 1 \ 2 \ \cdots \ p-1) \in \text{Sym}(\mathbb{F}_p)$. For $0 \leq i \leq n-1$ define $\sigma_i \in \text{Sym}(\mathbb{F}_p^n)$ as follows:*

$$\sigma_i(\lambda_0, \lambda_1, \dots, \lambda_{n-1}) = \begin{cases} (\lambda_0, \dots, \lambda_i, \dots, \lambda_{n-1}) & \exists j < i : \lambda_j \neq 0 \\ (\lambda_0, \dots, \sigma(\lambda_i), \dots, \lambda_{n-1}) & \forall j < i : \lambda_j = 0 \end{cases}.$$

Then $\langle \sigma_0, \dots, \sigma_{n-1} \rangle$ is a copy of P_n in $\text{Sym}(\mathbb{F}_p^n)$. It acts transitively.

Proof. More generally, for $G \leq \text{Sym}(Y)$ and $S \leq \text{Sym}(X)$, the group $G \wr S = G^X \rtimes S$ is the following subgroup of $\text{Sym}(X \times Y)$: The action of $\pi \in S$ on $X \times Y$ is $(x_0, y) \mapsto (\pi(x_0), y)$, and the action of $(g_x)_{x \in X} \in G^X$ is $(x_0, y) \mapsto (x_0, g_{x_0}(y))$. This is indeed an action of $G \wr S$, since

$$\begin{aligned} \pi(g_x)_{x \in X}(x_0, y) &= \pi(x_0, g_{x_0}(y)) = (\pi(x_0), g_{x_0}(y)) \\ &= (g_{\pi^{-1}(x)})_{x \in X}(\pi(x_0), y) = (g_{\pi^{-1}(x)})_{x \in X} \pi(x_0, y). \end{aligned}$$

For $g \in G$ and $x \in X$ define $\delta_x(g) \in G^X$ by $(\delta_x g)_{x'} = \begin{cases} g & x = x' \\ \text{Id} & \text{otherwise} \end{cases}$. Then ${}^\pi \delta_x(g) = \delta_{\pi(x)}(g)$. So as $(g_x)_{x \in X} = \prod_{x \in X} \delta_x(g_x)$, we see: If S is transitive and $x_0 \in X$ then G^X is the normal closure of $\text{Im}(\delta_{x_0})$, and $G \wr S$ is generated by S and $\text{Im}(\delta_{x_0})$. We apply this to $P_n = C_p \wr P_{n-1}$ and use induction over n . Note in particular that σ_{n-1} is $\delta_{x_0}(\sigma)$ for $x_0 = (0, \dots, 0) \in \mathbb{F}_p^{n-1}$.

Transitive: More generally, if G and S are transitive, then so is $G \wr S$. \square

Example 3.2. Consider $P_3 = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ for $p = 3$. Then for example $15 = 1 \cdot 9 + 2 \cdot 3 + 0 \cdot 1 \in \{0, 1, \dots, 26\}$ corresponds to $(1, 2, 0) \in \mathbb{F}_3^3$. Hence

$$\begin{aligned} \sigma_0 &= (0 \ 9 \ 18)(1 \ 10 \ 19)(2 \ 11 \ 20)(3 \ 12 \ 21)(4 \ 13 \ 22)(5 \ 14 \ 23)(6 \ 15 \ 24) \cdot \\ &\quad (7 \ 16 \ 25)(8 \ 17 \ 26) \\ \sigma_1 &= (0 \ 3 \ 6)(1 \ 4 \ 7)(2 \ 5 \ 8) \\ \sigma_2 &= (0 \ 1 \ 2). \end{aligned}$$

Lemma 3.3. *All p^{n-1} conjugates of σ_{n-1} in P_n commute with each other.*

Proof. P_{n-1} has degree p^{n-1} , and in the isomorphism $P_n \cong C_p \wr P_{n-1}$ the P_{n-1} is generated by $\sigma_0, \dots, \sigma_{n-2}$, and the C_p by σ_{n-1} . \square

Remark 3.4. For $x \in P_n$, observe that ${}^x \sigma_{n-1} \in P_n$ moves $(\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{F}_p^n$ if and only if x sends $(0, \dots, 0) \in \mathbb{F}_p^n$ to $(\lambda_0, \dots, \lambda_{n-2}, \mu)$ for some $\mu \in \mathbb{F}_p$. So ${}^x \sigma_{n-1}$ is a p -cycle on those p points whose first $n-1$ coordinates coincide with those of $x(0, \dots, 0)$. One such value of x is $x = \sigma_0^{\lambda_0} \sigma_1^{\lambda_1} \dots \sigma_{n-2}^{\lambda_{n-2}}$; this lies in P_{n-1} , viewed as a subgroup of P_n via the isomorphism $P_n \cong C_p \wr P_{n-1}$.

Notation 3.5. Following Weir we write A^{n-1} for the base group $C_p^{p^{n-1}}$ of $P_n = C_p \wr P_{n-1} = C_p^{p^{n-1}} \rtimes P_{n-1}$. Then A^{n-1} is elementary abelian, and normal in P_n . Also, A^{n-1} is the normal closure of $\langle \sigma_{n-1} \rangle$ in P_n .

Remark 3.6. For odd p , Weir [15, Thm 6] shows that A^{n-1} is the unique maximal abelian normal subgroup of P_n , and hence characteristic. But if $p = 2$ then A^{n-1} is not characteristic: for example, $P_2 \cong D_8$ and A^1 is one of the two rank two elementary abelians in D_8 ; but these two elementary abelians are conjugate in D_{16} .

Example 3.7. For $p = 3$, $A^2 \leq P_3$ is elementary abelian of rank 9 with basis

$$\begin{aligned} \sigma_2 &= (0 \ 1 \ 2) & \sigma_1 \sigma_2 &= (3 \ 4 \ 5) & \sigma_1^2 \sigma_2 &= (6 \ 7 \ 8) \\ \sigma_0 \sigma_2 &= (9 \ 10 \ 11) & \sigma_0 \sigma_1 \sigma_2 &= (12 \ 13 \ 14) & \sigma_0 \sigma_1^2 \sigma_2 &= (15 \ 16 \ 17) \\ \sigma_0^2 \sigma_2 &= (18 \ 19 \ 20) & \sigma_0^2 \sigma_1 \sigma_2 &= (21 \ 22 \ 23) & \sigma_0^2 \sigma_1^2 \sigma_2 &= (24 \ 25 \ 26). \end{aligned}$$

Corollary 3.8. *Both A^{n-1} and P_n are self-centralizing¹ in S_{p^n} .*

Proof. By Remark 3.4, A^{n-1} is generated by a set X of p -cycles whose supports are disjoint and cover \mathbb{F}_p^n . Suppose that $\pi \in S_{p^n}$ centralizes A^{n-1} , and pick $\sigma \in X$; then $[\pi, \sigma] = 1$. Since σ is a p -cycle and C_p is self-centralizing in S_p , it follows that π has the form $\pi = \pi' \cdot \sigma^r$, where the supports of π' and σ are disjoint. As the supports of the $\sigma \in X$ cover \mathbb{F}_p^n , it follows that $\pi \in A^{n-1}$. So A^{n-1} is self-centralizing in S_{p^n} , and the result for P_n follows. \square

¹We say that H is self-centralizing in G if $C_G(H) \leq H$.

4. WEIR'S FILTRATION T_j AND PROPOSITION 1.10

Notation 4.1. Associativity implies that $P_n \cong P_{n-j} \wr P_j$ for all $0 \leq j \leq n$. Weir [15] writes T_j for the base group of this wreath product, so $T_j \cong P_{n-j}^{p^j}$.

Hence $P_n = P_j T_j$, $T_{n-1} = A^{n-1}$ and $P_n = T_0 \geq T_1 \geq \cdots \geq T_{n-1} \geq T_n = 1$. Also, T_{j-1}/T_j is the subgroup A^{j-1} of $P_j \cong P_n/T_j$. For odd p this means that each T_j is characteristic in P_n , as A^{n-1} is characteristic.

Example 4.2. If $p^n = 3^3$ then $T_0 = P_3$; $T_3 = 1$; $T_2 = A^2$, which we described in Example 6.2; and T_1/T_2 is elementary abelian of rank 3, generated by the cosets of the three $\langle \sigma_0 \rangle$ -conjugates of σ_1 .

Notation 4.3. Weir defines the *depth* j of a subgroup $S \leq P_n$ to be the largest i such that $S \leq T_i$. That is, T_j is the smallest group in the series $P_n = T_0 > T_1 > \cdots > T_n = 1$ which contains S .

Lemma 4.4. *Let $N \trianglelefteq P_n$. If $N \cap T_{j+1} \not\leq N \cap T_j$ then $N \cap T_{k+1} \not\leq N \cap T_k$ for all $j \leq k \leq n-1$.*

Proof. If $g \in N \cap (T_j \setminus T_{j+1})$ then $gT_{j+1} \neq 1$ in the elementary abelian subgroup A^j of $P_n/T_{j+1} \cong P_{j+1}$. Replacing g by a conjugate, we may assume that σ_j lies in the support² of gT_{j+1} . Then $[g, \sigma_{j+1}] \in N \cap (T_{j+1} \setminus T_{j+2})$, since σ_{j+1} commutes with all nontrivial P_j -conjugates of σ_j . \square

Proof of Proposition 1.10. T_n and T_{n-1} are abelian. If $j < n-1$, then $N \cap T_{j+1}$ has depth $j+1$ by Lemma 4.4. So by downward induction on j we may assume that $[T_{j+1}, T_{j+1}] \leq N$.

T_j is generated by T_{j+1} and the P_j -conjugates of σ_j . So by the formulae³ for the commutators $[x, yz]$ and $[xy, z]$ it suffices to show that $[x, y] \in N$ if each of x, y is either an element of T_{j+1} or a P_j -conjugate of σ_j . As these conjugates commute with each other, we need only consider the case of $[\sigma_j, y]$, with $y \in T_{j+1}$.

If $[\sigma_j, y] = 1$ then we are done, hence we may assume that σ_j, y lie in the same factor $F \cong P_{n-j}$ of the base group of $P_{n-j} \wr P_j$. As in the proof of Lemma 4.4 there is some $g \in N$ such that σ_j occurs in the support of $gT_{j+1} \in A^j$. That is, some power g^r has component $\sigma_j z$ in F , with $z \in T_{j+1}$. Hence $[\sigma_j, y] = [g^r z^{-1}, y]$. Using the commutator formulae again we have $[\sigma_j, y] \in N$. \square

Corollary 4.5. (see [5, Thm 4.4.1]) *Let p be an arbitrary prime. If $B \leq P_n$ is an abelian normal subgroup, then $B \leq T_{n-2}$.*

Proof. If not, then $T'_{n-3} \leq B$ by Proposition 1.10. But T_{n-3} is a direct product of copies of P_3 , and P'_3 is nonabelian as $[[\sigma_0, \sigma_1], [\sigma_0, \sigma_2]] \neq 1$. \square

5. UNISERIAL ACTION AND PROPOSITION 1.8

Definition 5.1. Let P, M be finite p -groups, with M abelian and P acting on M . Recall from [13, §4.1] that the action is called *uniserial* if the following equivalent conditions hold:

- (1) $[P, N]$ has index p in N for every P -invariant subgroup $1 \neq N \leq M$.
- (2) $M_\ell \neq 0$, where $\ell = \log_p(|M|)$, $M_1 = M$ and $M_{r+1} = [P, M_r]$.

² A^j is an \mathbb{F}_p -vector space, with basis the P_j -conjugates of σ_j .

³See e.g. [8, Lemma 2.2.4, p. 20].

Recall further that if the action is uniserial, then

- (1) $M = M_1 > M_2 > \cdots > M_\ell > M_{\ell+1} = 0$.
- (2) $N \leq M$ is P -invariant if and only if N is one of the M_r .
- (3) The set of P -invariant subgroups of M is linearly ordered by inclusion.

One calls ℓ the *length* of M .

Remark 5.2. It follows that $C_M(P) = M_\ell$.

Lemma 5.3. *Let P, M be 2-groups, with P acting on M . Then the natural action of $Q = P \wr C_2$ on $M^2 = M \oplus M$ has the following properties:*

- (1) *If $[P, M]$ has index 2 in M , then $M^2 > [Q, M^2] > [Q, Q, M^2] = [P, M]^2$.*
- (2) *If the action of P on M is uniserial, then so is the action of Q on M^2 .*

Proof. Here we write $[a, b] = aba^{-1}b^{-1}$ and $[a, b, c] = [a, [b, c]]$.

1): Since the action of Q on M^2 is nilpotent and $[P, M]^2$ has index 4 in M^2 , it suffices to show that $[P, M]^2 \leq [Q, Q, M^2]$. We have $Q = P^2 \rtimes \langle \sigma \rangle$, where σ transposes the two copies of P^2 . Then for $g \in P$ and $x \in M$ we have

$$[(g, 1), \sigma, (1, x)] = [(g, 1), (x, x^{-1})] = ([g, x], 1)$$

Hence $[P, M] \times 1$ lies in $[Q, Q, M^2]$. Similarly, $1 \times [P, M] \leq [Q, Q, M^2]$.

2): Set $\ell = \log_2 |M|$. Define M_r, N_r by $M_1 = M$, $N_1 = M^2$, $M_{r+1} = [P, M_r]$ and $N_{r+1} = [Q, N_r]$. Then $M_\ell \neq 0$, and we need $N_{2\ell} \neq 0$. As M is uniserial, $M_{r+1} = [P_{n-1}, M_r]$ has index 2 in M_r for $r \leq \ell$. So by induction on r we have $N_{2r-1} = M_r^2$ for $r \leq \ell + 1$, since if $r \leq \ell$ and $N_{2r-1} = M_r^2$ then

$$N_{2r+1} = [Q, Q, M_r^2] = [P, M_r]^2 = M_{r+1}^2$$

by 1). In particular $N_{2\ell-1} = M_\ell^2 \neq 0$. Another application of 1) shows that $N_{2\ell} = [Q, M_\ell^2] > [Q, Q, M_\ell^2]$, hence $N_{2\ell} \neq 0$. So M^2 is uniserial. \square

Lemma 5.4 (Weir). *The action of P_n on A^n is uniserial of length p^n .*

Proof. By [15, Theorem 2] we need only consider the case $p = 2$. For $n = 1$ this is immediate; and for $n \geq 2$ it follows from Lemma 5.3 2) by induction on n , as the action of P_n on A^n is the induced action of $P_{n-1} \wr C_2$ on $(A^{n-1})^2$. \square

Proof of Proposition 1.8. For odd p we may take $B = A^{n-1}$ by Theorems 2 and 6 of Weir's paper [15], so from now on we take $p = 2$. Corollary 4.5 tells us that if $N \trianglelefteq P_n$ is abelian then $N \leq T_{n-2}$. Now T_{n-2} is the base group of $P_2 \wr P_{n-2}$, and $P_2 \cong D_8$: so $T_{n-2} \cong (D_8)^{2^{n-2}}$; and $P_n \cong T_{n-2} \rtimes P_{n-2}$, where P_{n-2} acts by permuting the copies of D_8 transitively.

Since N is abelian, its projection onto each D_8 must be abelian too; and since P_{n-2} acts transitively, each projection must be the same abelian subgroup of D_8 . But D_8 only has one abelian subgroup of exponent 4. So if N has exponent four then it is contained in $B = (C_4)^{2^{n-2}}$. Since B is normal in P_n it follows that B is the unique largest abelian normal subgroup of exponent four, and hence characteristic in P_n .

We have $P_n/B \cong (D_8/C_4)^{2^{n-2}} \rtimes P_{n-2} \cong C_2^{2^{n-2}} \rtimes P_{n-2} \cong P_{n-1}$. Writing B^{n-1} for B , we see that $P_1 \cong D_8/C_4$ acts uniserially on $B^1 \cong C_4$; and that $B^{n-1} \cong (B^{n-2})^2$ in $P_n \cong P_{n-1} \wr C_2$. So the action of P_{n-1} on B^{n-1} is uniserial by Lemma 5.3 2). \square

6. THE HALL SUBGROUP AND PROPOSITION 1.3

The group of units \mathbb{F}_p^\times is cyclic of order $p-1$: let r be a generator. Define $\eta \in \text{Sym}(\mathbb{F}_p)$ by $\eta(x) = rx$. Then η is a $(p-1)$ -cycle, with $\eta(0) = 0$. Since the σ of Lemma 3.1 is given by $\sigma(x) = x+1$, we have $\eta\sigma = \sigma^r$.

Lemma 6.1. For $0 \leq i \leq n-1$ define $\eta_i \in \text{Sym}(\mathbb{F}_p^n)$ by

$$\eta_i(\lambda_0, \lambda_1, \dots, \lambda_{n-1}) = (\lambda_0, \dots, \eta(\lambda_i), \dots, \lambda_{n-1}).$$

and set $H = \langle \eta_0, \dots, \eta_{n-1} \rangle$. Then $H \cong (C_{p-1})^n$, and $H \leq N_{S_{p^n}}(P_n)$.

Corollary 6.4 below shows that H is a Hall p' -subgroup of $N_{S_{p^n}}(P_n)$.

Proof. $H \cong (C_{p-1})^n$ is clear. And $\eta_j \sigma_i = \begin{cases} \sigma_i^r & j = i \\ \sigma_i & j \neq i \end{cases}$, since $\eta(0) = 0$. □

Example 6.2. For $p^n = 3^3$ we have $r = 2$ and

$$\begin{aligned} \eta_0 &= (9\ 18)(10\ 19)(11\ 20)(12\ 21)(13\ 22)(14\ 23)(15\ 24)(16\ 25)(17\ 26) \\ \eta_1 &= (3\ 6)(4\ 7)(5\ 8)(6\ 9) \cdot (12\ 15)(13\ 16)(14\ 17) \cdot (21\ 24)(22\ 25)(23\ 26) \\ \eta_2 &= (1\ 2)(4\ 5)(7\ 8)(10\ 11)(13\ 14)(16\ 17)(19\ 20)(22\ 23)(25\ 26). \end{aligned}$$

We are now in a position to prove Proposition 1.3.

Remark 6.3. The following observation follows from the fact that every submodule of \mathbb{Z}^n is free of rank $\leq n$: A finite abelian group G is isomorphic to a subgroup of $(C_m)^n$ if and only if the exponent of G divides m , and G has a generating set of size $\leq n$.

Proof of Proposition 1.3. We show that $\text{Aut}(P_n)$ has a normal Sylow p -subgroup Q , and an abelian Hall p' -subgroup A with exponent dividing $p-1$ and at most n generators; the result follows by Corollary 3.8 and Lemma 6.1.

It is well known that $\text{Aut}(C_p) \cong C_{p-1}$, see e.g. [8, Thm 1.3.10, p. 12]. That deals with the case $n = 1$, so now take $n \geq 2$.

Step 1: The subgroups B_i and the map ϕ .

Proposition 1.8 says that P_n has a characteristic abelian subgroup B such that $P_n/B \cong P_{n-1}$ acts uniserially on B . Define B_i inductively for $i \geq 0$ by $B_0 = B$ and $B_{i+1} = [P_n, B_i]$. Then each B_i is characteristic in P_n ; $B_i \leq B_{i-1}$; and the factor group B_{i-1}/B_i is cyclic of order p for all $i \leq p^{n-1}$, and $B_{p^{n-1}} = 1$.

As each term is characteristic in P_n , the normal series $P_n > B = B_0 > B_1 > \dots > B_{p^{n-1}} = 1$ induces

$$\phi: \text{Aut}(P_n) \rightarrow \text{Aut}(P_n/B) \times \prod_{i=1}^{p^{n-1}} \text{Aut}(B_{i-1}/B_i),$$

Step 2: $\text{Aut}(P_n)$ has a normal Sylow p -subgroup Q , and an abelian Hall p' -subgroup A of exponent dividing $p-1$.

The kernel of ϕ is a p -group by [8, Cor 5.3.3, p. 179]. Since $P_n/B \cong P_{n-1}$ and $\text{Aut}(B_{i-1}/B_i) \cong \text{Aut}(C_p) \cong C_{p-1}$, our ϕ is a map

$$\phi: \text{Aut}(P_n) \rightarrow \text{Aut}(P_{n-1}) \times (C_{p-1})^{p^{n-1}}.$$

By induction, $\text{Aut}(P_{n-1})$ has a normal Sylow p -subgroup whose factor group is abelian of exponent dividing $p-1$. Now apply Remark 1.7.

Step 3: The kernel K of $A \hookrightarrow \text{Aut}(P_n) \rightarrow \text{Aut}(P_n/B)$ is cyclic.

Suppose that $\alpha \in K$ acts trivially on B_{i-1}/B_i for some i . From $B_i = [P_n, B_{i-1}]$ it follows that B_i/B_{i+1} is generated by elements of the form $[g, x]B_{i+1}$, with $x \in B_{i-1}$ and $g \in P_n$. Then $\alpha([g, x]) = [\alpha(g), \alpha(x)]$. Since α acts trivially on B_{i-1}/B_i , we have $\alpha(x) = xy$ for some $y \in B_i$; and since $\alpha \in K$ we have $\alpha(g) = gz$ for some $z \in B$. So $\alpha([g, x]) = [gz, xy] = [g, xy] \in [g, x] \cdot [P_n, B_i] = [g, x]B_{i+1}$. So α acts trivially on B_i/B_{i+1} too. Hence: if $\alpha \in K$ acts trivially on B_0/B_1 then it acts trivially on each B_{i-1}/B_i , meaning that $\alpha \in \ker(\phi) \subseteq Q$. But $A \cap Q = 1$, so K acts faithfully on $B_0/B_1 \cong C_p$ and is therefore cyclic.

Step 4: A has at most n generators.

K is cyclic, and A/K is isomorphic to a p' -subgroup of $\text{Aut}(P_{n-1})$. By induction and Remark 1.7, A/K is isomorphic to a subgroup of C_{p-1}^{n-1} and has at most $n-1$ generators. So A has at most n generators. \square

Corollary 6.4. *The group H constructed in Lemma 6.1 is a Hall p' -subgroup of $N_{S_{p^n}}(P_n)$, and its image in $\text{Aut}(P_n)$ is a Hall p' -subgroup of $\text{Aut}(P_n)$.*

Proof. By Proposition 1.3 it has the correct order. \square

7. DIRECT SUMMANDS OF M^n FOR UNISERIAL M

In this section we take P to be a finite p -group.

Lemma 7.1. *Let $M \neq 0$ be a uniserial $\mathbb{F}_p P$ -module. Then there is some $a_0 \in \mathbb{F}_p P$ with $a_0 M = C_M(P)$ and $a_0[P, M] = 0$.*

Proof. Let $I = \{a \in \mathbb{F}_p P \mid aM = 0\}$, the annihilator of M in $\mathbb{F}_p P$. Observe that I is a two-sided ideal in $\mathbb{F}_p P$, and proper since $M \neq 0$. Hence the quotient ring $R = \mathbb{F}_p P/I$ has order p^d for some $d \geq 1$. Now, the p -group $P \times P$ acts on R via $(x, y) \cdot (r + I) = xry^{-1} + I$; and so the number of length one orbits has to be divisible by p . As $0 + I$ is one such orbit, it follows that $a_0 + I$ is fixed by $P \times P$ for some $a_0 \notin I$. Then for all $g, h \in P$ and all $x \in M$ we have $ga_0hx = a_0x$. Hence $a_0 M$ is a submodule of $C_M(P)$; and since $[P, M]$ is generated by elements of the form $(h-1)x$, it follows that $a_0[P, M] = 0$. Moreover, since $a_0 \notin I$ we have $a_0 M \neq 0$. But since M is uniserial it follows that $C_M(P)$ is simple, so $a_0 M = C_M(P)$. \square

Lemma 7.2. *Let P be a p -group and M a length ℓ uniserial $\mathbb{F}_p P$ -module. Let $N_v \subseteq M^n$ be the cyclic submodule generated by $v = (v_1, \dots, v_n) \in M^n$. Then the following statements are equivalent:*

- (1) *As an $\mathbb{F}_p P$ -module, N_v is uniserial of length ℓ .*
- (2) *$\dim_{\mathbb{F}_p}(N_v) = \ell$ and $\dim_{\mathbb{F}_p}(C_{N_v}(P)) = 1$.*
- (3) *There is some $i \in \{1, \dots, n\}$ with the following properties:*
 - (a) *If $a \in \mathbb{F}_p P$ satisfies $av_i = 0$, then $av = 0$.*
 - (b) *$v_i \in M$ lies outside $[P, M]$.*

Example 11.1 shows that we cannot dispense with condition 3a).

Proof. 1) \Rightarrow 2): Follows from Remark 5.2.

2) \Rightarrow 3): Pick $0 \neq w \in C_{N_v}(P)$, then $w_i \neq 0$ for some $i \in \{1, \dots, n\}$. Now consider the $\mathbb{F}_p P$ -module map $\phi: N_v \rightarrow M$, $u \mapsto u_i$. If $u \in \ker(\phi)$, then the submodule $U \subseteq N_v$ generated by u satisfies $x_i = 0$ for all $x \in U$. If $U \neq 0$ then $C_U(P) \neq 0$ and therefore $U \cap C_{N_v}(P) \neq 0$. So there is $0 \neq w' \in U \cap C_{N_v}(P)$. As $w_i \neq 0$

and $w' \subseteq U \subseteq \ker(\phi)$, we see that w, w' are linearly independent, a contradiction. Hence ϕ is injective.

Since $av_i = \phi(av)$, this proves 3a). Also, since ϕ is injective, it is surjective for dimension reasons. So $v_i = \phi(v)$ generates M , since v generates N_v . This shows 3b), since $[P, M]$ is a proper submodule.

3) \Rightarrow 1): Conversely, 3a) means that ϕ is injective, and since M is uniserial, 3b) means that ϕ is surjective. So N_v is isomorphic to M . \square

Lemma 7.3. *Let $M \neq 0$ be a length ℓ uniserial $\mathbb{F}_p P$ -module; $v_1, \dots, v_r \in M^n$ elements satisfying the equivalent conditions of Lemma 7.2; and $N = \sum_{i=1}^r N_{v_i}$ the $\mathbb{F}_p P$ -submodule they generate. Then the following statements are equivalent:*

- (1) *The sum $N = \sum_{i=1}^r N_{v_i}$ is direct.*
- (2) *The images of v_1, \dots, v_r in $M^n/[P, M^n]$ are linearly independent.*
- (3) *If w_i is a basis element of $C_{N_{v_i}}(P)$, then w_1, \dots, w_r are linearly independent.*

Proof. 2) \Leftrightarrow 3): Let $a_0 \in \mathbb{F}_p P$ be as in Lemma 7.1. Then the map $w \mapsto a_0 w$ induces an isomorphism $M^n/[P, M^n] \rightarrow C_{M^n}(P)$. Up to multiplication by an invertible scalar we then have $w_i = a_0 v_i$, hence 2) \Leftrightarrow 3).

3) \Rightarrow 1): If $\sum_i u_i = 0$ with $u_i \in N_{v_i}$ and not all $u_i = 0$, then by nilpotence we get a linear dependence between the w_i .

1) \Rightarrow 3): If $\sum_i N_{v_i}$ is direct, then $\sum_i C_{N_{v_i}}(P)$ is direct too. \square

Lemma 7.4. *Let $M \neq 0$ be a length ℓ uniserial $\mathbb{F}_p P$ -module. For an $\mathbb{F}_p P$ -submodule N of M^n , the following four statements are equivalent:*

- (1) *N is a direct summand of M^n .*
- (2) *N has a generating set v_1, \dots, v_r satisfying the equivalent conditions of Lemma 7.3.*
- (3) *The \mathbb{F}_p -vector spaces $(N + [P, M^n])/[P, M^n]$ and $C_N(P)$ have the same dimension.*
- (4) *M_Z is a complement of N in M^n for some $Z \subseteq \{1, 2, \dots, n\}$. Here, $M_Z = \{(u_1, \dots, u_n) \in M^n \mid u_i = 0 \text{ for all } i \notin Z\}$.*

If these equivalent conditions hold then:

- (5) *For any complement L of N in M^n , the normal subgroup N of $M^n \rtimes P$ has complement $L \rtimes P$.*
- (6) *With r as in 2), we have $\dim C_N(P) = r$ in 3) and $|Z| = n - r$ in 4).*

Proof. 1) \Rightarrow 2): M^n is a direct sum of n copies of the length ℓ uniserial module M . By Krull-Schmidt, N is also a direct sum of length ℓ uniserial modules; and uniserial modules are cyclic.

2) \Rightarrow 3) and first part of 6): As $N = \bigoplus_{i=1}^r N_{v_i}$ we have $\dim C_N(P) = r$ since $\dim C_{N_{v_i}}(P) = 1$, and $\dim N/[P, N] = r$ since $\dim N_{v_i}/[P, N_{v_i}] = 1$.

3) \Rightarrow 4) and second part of 6): $C_N(P)$ is a subspace of $C_{M^n}(P)$. Pick $0 \neq w \in C_M(P)$, and define $w_i \in C_{M^n}(P)$ by $w_i = (0, \dots, 0, w, 0, \dots, 0) \in M^n$, with w in the i th position. Then w_1, \dots, w_n is a basis of $C_{M^n}(P)$, so by the exchange lemma there is $Z \subseteq \{1, \dots, n\}$ such that the subspace $W_Z \subseteq C_{M^n}(P)$ on the w_i with $i \in Z$ is a complement to $C_N(P)$. In particular, $|Z| = n - \dim C_N(P)$.

Since M_Z has socle W_Z and N has socle $C_N(P)$, it follows that the sum $M_Z + N$ is direct. Now suppose that $x \in M_Z$, $y \in N$ and $x + y \in [P, M^n]$. With a_0 as

in Lemma 7.1 we have $a_0x + a_0y = 0$. Since $M_Z + N$ is direct, it follows that $a_0x = a_0y = 0$, and hence $x, y \in [P, M^n]$. Therefore

$$\begin{aligned} \dim \frac{(M_Z \oplus N) + [P, M^n]}{[P, M^n]} &= \dim \frac{M_Z + [P, M^n]}{[P, M^n]} + \dim \frac{N + [P, M^n]}{[P, M^n]} \\ &= (n - \dim C_N(P)) + \dim C_N(P) = n. \end{aligned}$$

Hence $(M_Z \oplus N) + [P, M^n] = M^n$, so as $[P, _]$ is nilpotent we conclude that $M_Z \oplus N = M^n$.

Finally, 4) \Rightarrow 1) and 5) are clear. \square

8. UNISERIAL MODULES AND P_j

Remark 8.1. By Lemma 5.4 the natural action of P_n on $M = (\mathbb{F}_p)^{p^n}$ is uniserial of length p^n . Observe that the socle $C_M(P_n)$ is the diagonal subgroup

$$\Delta(\mathbb{F}_p) = \{\underline{v} \in (\mathbb{F}_p)^{p^n} \mid v_i = v_j \text{ for all } i, j\},$$

and that

$$[P_n, M] = \{\underline{v} \in (\mathbb{F}_p)^{p^n} \mid \sum_i v_i = 0\}.$$

Lemma 8.2. *Let $0 \leq j \leq n$. Set $K := [T_j, T_j]$. For any $U \leq P_n$ write $\bar{U} = UK/K$. Then $K \leq T_j$, and*

- (1) *The quotient module $\bar{T}_j = T_j/K$ is an $\mathbb{F}_p P_j$ -module.*
- (2) *\bar{T}_j is the direct sum of $n - j$ length p^j uniserial modules isomorphic to A^j ; these summands are generated by $\sigma_j K, \sigma_{j+1} K, \dots, \sigma_{n-1} K$.*
- (3) *$C_{\bar{T}_j}(P_j)$ is the \mathbb{F}_p -vector space with basis $\Delta(\sigma_j)K, \dots, \Delta(\sigma_{n-1})K$.*
- (4) *If L is a direct summand of the $\mathbb{F}_p P_j$ -module \bar{T}_j , then there is some $Z \subseteq \{\sigma_j, \dots, \sigma_{n-1}\}$ such that $\bar{T}_j = L \oplus M_Z$, where $M_Z \subseteq \bar{T}_j$ is the submodule generated by $\{\sigma_i K \mid \sigma_i \in Z\}$.*

Proof. 1): In the factorization $P_n = P_j T_j$, note that $P_j = \langle \sigma_0, \dots, \sigma_{j-1} \rangle$, and that T_j is the normal closure of $\langle \sigma_j, \dots, \sigma_{n-1} \rangle$ under the action of P_j . Since each σ_i has order p , the abelianization of T_j is elementary abelian.

2): The submodule of \bar{T}_j generated by $\sigma_i K$ has basis consisting of the p^j conjugates of $\sigma_i K$ under the action of P_j . So \bar{T}_j is the direct sum of these submodules, and each is isomorphic to A^j . As we recalled in Remark 8.1, Weir showed that A^j is uniserial of length p^j .

3): $C_{\bar{T}_j}(P_j)$ is the diagonal subgroup by Remark 8.1.

4): Lemma 7.4, specifically 1) \Leftrightarrow 4). \square

For the next lemma we suppose that L is a direct summand of the $\mathbb{F}_p P_j$ -module \bar{T}_j . From Lemma 8.2 we know that $\bar{T}_j = L \oplus M_Z$ for some $Z \subseteq \{\sigma_j, \dots, \sigma_{n-1}\}$; and that $C_{\bar{T}_j}(P_j)$ has basis $\Delta(\sigma_j)K, \dots, \Delta(\sigma_{n-1})K$.

Lemma 8.3. *Under these circumstances we have:*

- (1) *If $L \not\subseteq \bar{T}_{j+1} + [P_j, \bar{T}_j]$ then we may choose Z such that $\sigma_j \notin Z$.*
- (2) *If $L + [P_j, \bar{T}_j] = \bar{T}_{j+1} + [P_j, \bar{T}_j]$ then $Z = \{\sigma_j\}$.*
- (3) *If $L + [P_j, \bar{T}_j] \subsetneq \bar{T}_{j+1} + [P_j, \bar{T}_j]$ then for every complement D of L there are $x, y \in C_D(P_j)$ such that $\Delta(\sigma_j)K$ lies in the support of x , whereas the support of $y \neq 0$ does not contain $\Delta(\sigma_j)K$.*

Proof. Let D be a complement of L . By Lemma 7.1 there is some $a_0 \in \mathbb{F}_p P_j$ such that $a_0 \bar{T}_j = C_{T_j}(P_j)$, and hence

$$C_{\bar{T}_j}(P_j) = C_L(P_j) \oplus C_D(P_j).$$

Moreover, multiplication by a_0 induces an isomorphism $\frac{\bar{T}_j}{[P_j, \bar{T}_j]} \xrightarrow[\cong]{\mu} C_{\bar{T}_j}(P_j)$ which restricts to isomorphisms

$$\frac{L + [P_j, \bar{T}_j]}{[P_j, \bar{T}_j]} \xrightarrow{\cong} C_L(P_j) \quad \text{and} \quad \frac{D + [P_j, \bar{T}_j]}{[P_j, \bar{T}_j]} \xrightarrow{\cong} C_D(P_j).$$

- 1): Recall from the proof of Lemma 7.4 that Z is chosen using the exchange lemma: Z is any subset of $\{\sigma_j, \dots, \sigma_{n-1}\}$ such that $\{\Delta(\sigma_i)K \mid \sigma_i \in Z\}$ is the basis of a complement to $C_L(P_j)$. Now, $\frac{\bar{T}_{j+1} + [P_j, \bar{T}_j]}{[P_j, \bar{T}_j]}$ is precisely the preimage under μ of the subspace spanned by $\Delta(\sigma_{j+1})K, \dots, \Delta(\sigma_{n-1})K$. So by assumption there is some $x \in L$ such that $a_0 x$ has $\Delta(\sigma_j)K$ in its support. Beginning the exchange lemma with x , we can ensure that $\sigma_j \notin Z$.
- 2): $C_L(P_j)$ has basis $\Delta(\sigma_{j+1})K, \dots, \Delta(\sigma_{n-1})K$, hence $Z = \{\sigma_j\}$.
- 3): $C_L(P_j)$ is a proper subspace of the span of $\Delta(\sigma_{j+1})K, \dots, \Delta(\sigma_{n-1})K$. The result follows, as $C_D(P_j)$ is a complement in $C_{\bar{T}_j}(P_j)$. \square

9. COMPLEMENTS AND UNISERIAL MODULES

We recall some notation from Lemma 8.2: So $K = [T_j, T_j]$, and $\bar{U} = UK/K$.

Lemma 9.1. *Suppose that $N \trianglelefteq P_n$ has a complement C . Set $D = T_j \cap C$, where j is the depth of N . Then*

- (1) D is elementary abelian, and $\bar{D} \cong D$.
- (2) \bar{D} is an $\mathbb{F}_p P_j$ module, and $\bar{T}_j = \bar{N} \oplus \bar{D}$.

Proof. 1): Since $D \leq T_j$, $D \cap N = 1$ and $K \leq N$, the map from T_j to its abelianization $\bar{T}_j = T_j/K$ is injective on D . By Lemma 8.2, the abelianization is elementary abelian.

2): D is a group-theoretic complement of N in T_j , but it is conceivable that it is not normalized by P_j . However, if $a \in P_j$ then $a = cn$ with $c \in C$ and $n \in N$, so for $d \in D$ we have ${}^a d = {}^c(d \cdot d^{-1} n d n^{-1}) \in {}^c(DK) = DK$. Hence ${}^a \bar{D} = \bar{D}$. \square

Lemma 9.2. *Let Q be a p -group and $P = Q \wr C_p$, so $P = B \rtimes C_p$ with $B = Q^p$. Suppose that $x \in P \setminus B$ and $y \in B \cap C_P(x)$. If $y^p = 1$ then $y \in P'$.*

Proof. We have $x = (q_0, \dots, q_{p-1})\sigma$, with σ a p -cycle. Rearranging the factors in $B = Q^p$ if necessary, we may assume that $\sigma = (0 \ 1 \ 2 \ \dots \ p-2 \ p-1)$, so $\sigma(i) = i+1$ modulo p .

Special case: Q abelian: Since y lies in B it has the form $y = (q'_0, \dots, q'_{p-1})$. As Q is abelian we have ${}^x y = \sigma y = (q'_{p-1}, q'_0, q'_1, \dots, q'_{p-2})$; and so since $y \in C_P(x)$ it follows that $y = (q, q, \dots, q)$ for some $q \in Q$. As $y^p = 1$ we have $q^p = 1$. Now set $z = (1, q, q^2, \dots, q^{p-1}) \in Q^p$, then $\sigma z = (q, q^2, \dots, q^{p-1}, 1) = (q, q^2, \dots, q^p) = yz$. So $y = \sigma z \cdot z^{-1} \in P'$.

General case: We have $B' = (Q')^p$, so $P/B' \cong (Q/Q') \wr C_p$ and $yB' \in (P/B')'$ by the special case. Hence $y \in P'$. \square

Lemma 9.3. *Suppose that $N \trianglelefteq P_n$ has depth j . If N has a complement in P_n , then $\bar{N} + [P_j, \bar{T}_j]$ is a not proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$.*

Proof. Call the complement C , and set $D = C \cap T_j$. Lemma 9.1 says that the $\mathbb{F}_p P_j$ -module \bar{D} is a complement of \bar{N} in \bar{T}_j . If $\bar{N} + [P_j, \bar{T}_j]$ is a proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$ then Lemma 8.3 says that there are $x, y \in D$ such that the support of $xK \in C_{\bar{D}}(P_j)$ contains $\Delta(\sigma_j)K$, and yK is a nontrivial element of $C_{\bar{T}_{j+1}}(P_j)$.

Lemma 9.1 says that $\langle x, y \rangle$ is elementary abelian. On the other hand, $x, y \in T_j \cong (P_{n-j})^{p^j}$. Let $x_i, y_i \in P_{n-j}$ be the images of x, y in the i th of these p^j factors, then $\langle x_i, y_i \rangle$ is elementary abelian too. Moreover, our choice of x means that each x_i lies outside the base subgroup $(P_{n-j-1})^p$ of $P_{n-j} = P_{n-j-1} \wr C_p$; but y_i does lie in this base group, since $y \in T_{j+1}$. So $y_i \in P'_{n-j} \leq K \leq N$ by Lemma 9.2 and Proposition 1.10. As y is the product of the y_i , we have $y \in N \cap D = 1$: a contradiction. \square

Lemma 9.4. *Suppose that $N \trianglelefteq P_n$ has depth j . Then*

- (1) *If $K \leq L \leq T_j$, then $LP'_n/P'_n \cong \frac{\bar{L} + [P_j, \bar{T}_j]}{[P_j, \bar{T}_j]}$.*
- (2) *$NP'_n/P'_n = T_{j+1}P'_n/P'_n$ if and only if $\bar{N} + [P_j, \bar{T}_j] = \bar{T}_{j+1} + [P_j, \bar{T}_j]$.*
- (3) *NP'_n/P'_n is a proper subgroup of $T_{j+1}P'_n/P'_n$ if and only if $\bar{N} + [P_j, \bar{T}_j]$ is a proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$.*

Proof. By Proposition 1.10, $K = [T_j, T_j]$, satisfies $K \leq N \cap T_{j+1}$. Since $P_n = T_j \cdot P_j$ we have $P'_n = K \cdot [P_j, T_j] \cdot P'_j$ and therefore

$$P'_n \cap T_j = K \cdot [P_j, T_j].$$

- 1): $L \cap P'_n = L \cap (P'_n \cap T_j)$, and so $LP'_n/P'_n \cong L(P'_n \cap T_j)/(P'_n \cap T_j)$. The result follows, since $L(P'_n \cap T_j) = LK[P_j, T_j] = L[P_j, T_j]$.
- 2) and 3): Follow by applying 1) to the cases $L = N$ and $L = T_{j+1}$. \square

10. THE PERMUTATIONS ρ_i AND THEOREM 1.5

Notation 10.1. Now suppose that $1 \leq i \leq n-1$. Observe that $\langle \sigma_{i-1}, \sigma_i \rangle \cong P_2$, whose centre is cyclic of order p and generated by $\prod_{s=0}^{p-1} \sigma_{i-1}^s \sigma_i$. Set

$$\rho_i = \prod_{s=1}^{p-1} \sigma_{i-1}^s \sigma_i,$$

so $\sigma_i \rho_i$ generates the centre of $\langle \sigma_{i-1}, \sigma_i \rangle$.

- Lemma 10.2.**
- (1) $\rho_i^p = 1$.
 - (2) $\rho_i [T_j, T_j] = \sigma_i^{-1} [T_j, T_j] \forall i > j$.
 - (3) *Every P_j -conjugate of ρ_i commutes with every P_j -conjugate of ρ_k , for all $i, k > j$.*
 - (4) ${}^{\eta_k} \rho_k = \rho_k^r$, and ${}^{\eta_k} \rho_i = \rho_i$ for all $i \neq k$.

Example 10.3. If $p^n = 3^3$ then

$$\begin{aligned} \rho_1 &= (9 \ 12 \ 15)(10 \ 13 \ 16)(11 \ 14 \ 17)(18 \ 21 \ 24)(19 \ 22 \ 25)(20 \ 23 \ 26) \\ \rho_2 &= (3 \ 4 \ 5)(6 \ 7 \ 8). \end{aligned}$$

Proof. 1): The σ_{i-1} -conjugates of σ_i commute with each other, and each has exponent p . 2): $\sigma_i^p = 1$, and for $i > j$ have $\sigma_{i-1}^k \sigma_i \in \sigma_i [T_j, T_j]$.

3): Let $i > j$ and $\pi \in P_j$. Then $\pi \rho_i$ only alters $(\lambda_0, \dots, \lambda_{n-1})$ if $\lambda_{i-1} \neq 0$; $\lambda_k = 0$ for $j \leq k < i-1$; and $(\lambda_0, \dots, \lambda_{j-1}) = \pi(0, \dots, 0)$. If these conditions hold, then the value of λ_i is increased by 1. Any two such permutations commute with each other. 4): By inspection. \square

Proposition 10.4. *Suppose that $N \trianglelefteq P_n$. Set $j := \text{depth}(N)$. Then $K := T'_j \leq N$, and the following statements are equivalent:*

- (1) N has a complement in P_n .
- (2) N has an H -invariant complement in P_n .
- (3) N/K is a direct summand of the $\mathbb{F}_p P_j$ -submodule T_j/K , and NP'_n/P'_n is not a proper subgroup of $T_{j+1}P'_n/P'_n$.

Remark 10.5. As T_j/K is a direct sum of copies of the length p^j uniserial module A^j , one may use the equivalent conditions of Lemma 7.4 in order to determine whether N/K is a direct summand.

Proof. Proposition 1.10 tells us that $K := [T_j, T_j] \leq N$. The implication $2) \Rightarrow 1)$ is clear. As in Lemma 8.2 we write $\bar{U} = UK/K$.

$1) \Rightarrow 3)$: Lemma 9.1 says that \bar{N} is a direct summand of \bar{T}_j , and Lemma 9.3 says that $\bar{N} + [P_j, \bar{T}_j]$ is a not proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$. So NP'_n/P'_n is not a proper subgroup of $T_{j+1}P'_n/P'_n$ by Lemma 9.4.

$3) \Rightarrow 2)$: By Lemma 9.4, $\bar{N} + [P_j, \bar{T}_j]$ is a not proper subgroup of $\bar{T}_{j+1} + [P_j, \bar{T}_j]$. So we are in one of the first two cases of Lemma 8.3.

Suppose case 1) applies. Let $D \leq T_j$ be the subgroup generated by all P_j -conjugates of the ρ_i for which $\sigma_i \in Z$. Lemma 10.2 says that D is elementary abelian, and by construction it is normalized by P_j . Moreover, the formula for $\eta \sigma_i$ in the proof of Lemma 6.1 shows that P_j is H -invariant. So from Lemma 10.24) we conclude that D and $C := D \rtimes P_j$ are H -invariant.

From $\sigma_j \notin Z$ and Lemma 10.22) it follows that \bar{D} is M_Z , which is a complement of \bar{N} in \bar{T}_j . Moreover, D and M_Z are elementary abelian of the same rank, so $D \cap K = 1$ and $D \cap N \cong \bar{D} \cap \bar{N} = 1$. Hence $C \cap N = (T_j \cap C) \cap N = D \cap N = 1$, and C is a complement of N in P_n .

Now suppose case 2) applies. Let $D \leq T_j$ be the subgroup generated by all P_j -conjugates of σ_j . Then D is elementary abelian, $D \cap K = 1$, and \bar{D} is a complement to \bar{N} in \bar{T}_j . Hence $C = D \rtimes P_j$ is a complement to N in P_n . \square

Proof of Theorem 1.5. If $p = 2$ then we may take $H = 1$ by Proposition 1.3. For odd p , Corollary 6.4 says that the subgroup $H \leq S_{p^n}$ of Lemma 6.1 is a Hall p' -subgroup of the normalizer of $P_n = \langle \sigma_0, \dots, \sigma_{n-1} \rangle$. The result for this H follows from Proposition 10.4. \square

11. EXAMPLES

Example 11.1. This example concerns Lemma 7.2: it demonstrates that 3a) does not follow from 3b). For $p^n = 3^2$ let M be the length 9 uniserial P_2 -module $M = (\mathbb{F}_3)^9$. Consider $v = (v_1, v_2) \in M^2$ given by

$$v_1 = (1, 0, 0, 0, 0, 0, 0, 0, 0) \quad v_2 = (0, 0, 0, -1, 1, 0, 0, 0, 0).$$

Note that $v_1 \notin [P_2, M]$ and $v_2 \in [P_2, M]$: so 3b) is satisfied. Setting

$$a = \sigma_1 - \text{Id} = (0 \ 1 \ 2) - \text{Id} \in \mathbb{F}_3 P_2 \quad b = \sigma_0 a = (3 \ 4 \ 5) - \text{Id} \in \mathbb{F}_3 P_2$$

we see that v fails to satisfy 3a), since

$$\begin{aligned} av_1 &= (-1, 1, 0, 0, 0, 0, 0, 0) & av_2 &= \underline{0} \\ bv_1 &= \underline{0} & bv_2 &= (0, 0, 0, 1, 1, 1, 0, 0). \end{aligned}$$

Since $0 \neq av \in M \oplus 0$ and $0 \neq bv \in 0 \oplus M$, it follows that $\text{soc}(M^2) \subseteq M_v$. So 2) is also violated, as $C_{M^2}(P_2) = \text{soc}(M^2)$; and since $\text{soc}(M^2)$ has three separate dimension one submodules, M_v is not uniserial either, i.e. 1) is violated too.

The remaining examples concern normal subgroups of P_n .

Remark 11.2. We need a method for determining whether N has a complement, and constructing an H -invariant complement C if there is one.

The proof of 3) \Rightarrow 2) in Proposition 10.4 can readily be adapted for this purpose. First one checks whether \bar{N} is a direct summand of \bar{T}_j , possibly using the equivalent conditions of Lemma 7.4. If \bar{N} is a direct summand, then there are three possibilities:

- (1) If \bar{N} has a complement of the form M_Z with $Z \subseteq \{\sigma_{j+1}, \dots, \sigma_{n-1}\}$ then we take $C = \langle X \rangle$ for $X = \{\sigma_0, \dots, \sigma_{j-1}\} \cup \{\rho_i \mid \sigma_i \in Z\}$. This is the case $\sigma_j \notin Z$ and $N \not\leq T_{j+1}P'_n$.
- (2) If $NP'_n/P'_n = T_{j+1}P'_n/P'_n$ then \bar{N} has complement M_Z for $Z = \{\sigma_j\}$. We take $C = \langle \sigma_0, \dots, \sigma_j \rangle$.
- (3) If $NP'_n/P'_n \leq T_{j+1}P'_n/P'_n$ then N has no complement in P_n .

Sometimes it may be better to begin by comparing NP'_n/P'_n and $T_{j+1}P'_n/P'_n$.

Example 11.3. On why ρ_i replaces σ_i in Case 1) of Remark 11.2.

For $p^n = 3^3$ let N be the normal closure of $\langle \sigma_0 \rangle$ in P_3 . Then $j = 0$, so $K = P'_3$ and \bar{T}_0 is the \mathbb{F}_3 -vector space with basis $\sigma_0K, \sigma_1K, \sigma_2K$. As \bar{N} has basis σ_0K , it has complement M_Z for $Z = \{\sigma_1, \sigma_2\}$. Case 1) of Remark 11.2 says that $C = \langle \rho_1, \rho_2 \rangle$ is an H -invariant complement of N in P_3 .

Since this complement is elementary abelian, it has order 3^2 . By contrast, $\langle \sigma_1, \sigma_2 \rangle \cong P_2$ has order 3^4 . Hence $\langle \sigma_1, \sigma_2 \rangle \cong P_2$ is not a complement of N . In particular, $[\sigma_1, \sigma_2] \in \langle \sigma_1, \sigma_2 \rangle \cap N$, since $P'_3 = K \leq N$.

Example 11.4. This example features Case 2) of Remark 11.2. More significantly, it demonstrates that the normal subgroup N need not be H -invariant.

For $p^n = 3^3$ let N be the normal closure of $\langle \gamma\sigma_2 \rangle$ in P_3 , where $\gamma = \sigma_1 \cdot \sigma_0 \sigma_1 \cdot \sigma_0^2 \sigma_1$ is the product of the three $\langle \sigma_0 \rangle$ -conjugates of σ_1 .

Then N has depth $j = 1$, and $NP'_3/P'_3 = \langle \sigma_2 \rangle P'_3/P'_3 = T_2P'_3/P'_3$. We are in Case 2) of Remark 11.2 – provided that \bar{N} does have a complement.

\bar{T}_1 is a direct sum of two copies of the uniserial \mathbb{F}_3P_1 -module A^1 : one on σ_1K , and the other on σ_2K . Moreover, \bar{N} is generated by $v = \gamma K + \sigma_2K$. But γK lies in the socle of the summand on σ_1K , hence v satisfies Lemma 7.23) with $i = 2$. So v is a generating set for \bar{N} satisfying the conditions of Lemma 7.3, meaning that \bar{N} is a direct summand of \bar{T}_1 by Lemma 7.4. We conclude that N does have an H -invariant complement in P_3 .

By Case 2), one H -invariant complement is $C = \langle \sigma_0, \sigma_1 \rangle$.

Observe that if $\pi_1 = \sigma_2, \pi_2, \dots, \pi_9$ are the nine P_3 -conjugates of σ_2 (see Example 3.7), then

$$N = \left\{ \gamma^{\sum_{i=1}^9 e_i} \prod_{i=1}^9 \pi_i^{e_i} \mid e_1, \dots, e_9 \in \mathbb{Z} \right\}.$$

So as η_2 fixes σ_0, σ_1 and inverts σ_2 , we have

$$\eta_2 N = \left\{ \gamma^{-\sum_{i=1}^9 e_i} \prod_{i=1}^9 \pi_i^{e_i} \mid e_1, \dots, e_9 \in \mathbb{Z} \right\} \neq N.$$

So the normal subgroups N and $\eta_2 N$ of P_3 fail to be H -invariant – and yet each of them has a H -invariant complement in C .

Example 11.5. This example features Case 3) of Remark 11.2.

As in Example 11.4 we let N be the normal closure of $\langle \gamma \sigma_2 \rangle$, but this time we take $p^n = 3^4$, and so N is the normal closure in P_4 . Once more, the depth is $j = 1$ and \bar{N} is uniserial of length 3 and hence a direct summand of \bar{T}_1 . However this time $NP'_4/P'_4 = \langle \sigma_2 \rangle P'_4/P'_4$ is a proper subgroup of $T_2 P'_4/P'_4 = \langle \sigma_2, \sigma_3 \rangle P'_4/P'_4$. So N does not have a complement in P_4 , even though \bar{N} is a direct summand of \bar{T}_1 .

Example 11.6. The H -invariant complement need not be unique. Also, the distinction between cases 1) and 2) of Remark 11.2 is slightly arbitrary: for $N \not\leq T_{j+1} P'_n$ there may be Z with $\bar{T}_j = \bar{N} \oplus M_Z$ and $\sigma_j \in Z$.

For $p^n = 3^2$ let $N \leq P_2$ be the normal closure of $\langle \sigma_0 \sigma_1 \rangle$. This has depth $j = 0$, so $\bar{T}_0 = P_2/P'_2$ is the \mathbb{F}_3 -vector space with basis $\sigma_0 K, \sigma_1 K$, and \bar{N} is the subspace spanned by $\sigma_0 K + \sigma_1 K$. So we may take $Z = \{\sigma_1\}$, obtaining the H -invariant complement $\langle \rho_1 \rangle$; or we may take $Z = \langle \sigma_0 \rangle$, obtaining the H -invariant complement $\langle \sigma_0 \rangle$. Observe that $\langle \sigma_1 \rangle$ is a third H -invariant complement.

Example 11.7. In this example, \bar{N} is not a direct summand of \bar{T}_j .

For $p^n = 3^4$ we let N be the normal closure of $\langle \beta \rangle$ in P_4 , for $\delta = \sigma_2 \cdot \sigma_0 (\sigma_3^{-1} \cdot \sigma_1 \sigma_3)$. So $j = 2$ and \bar{T}_2 is the direct sum of two copies of A^2 , which is uniserial of length 9; and one can verify that δK corresponds to the element v of Example 11.1. So \bar{N} is not a direct summand of \bar{T}_2 .

12. PARTITION SUBGROUPS

Remark 12.1. Following Weir [15, p. 537] we define A_i^{n-1} inductively for $i \geq 0$ by $A_0^{n-1} = A^{n-1}$ and $A_{i+1}^{n-1} = [P_n, A_i^{n-1}]$. Then each A_i^{n-1} is normal in P_n , whence $A_i^{n-1} \leq A_{i-1}^{n-1}$. By [15, Theorem 2], the factor group A_{i-1}^{n-1}/A_i^{n-1} is cyclic of order p for all $i \leq \log_p(|A^{n-1}|) = p^{n-1}$, and $A_{p^{n-1}}^{n-1} = 1$.

Since $P_n = A^{n-1} \rtimes P_{n-1}$, one may view A_i^j as a subgroup of P_n for all $0 \leq j \leq n-1$. Since A_i^{n-1} is normal in P_n , it follows that every product of the form $Q = A_{i_0}^0 A_{i_1}^1 \cdots A_{i_{n-1}}^{n-1}$ is a subgroup of P_n . Weir calls the subgroups of this form *partition subgroups* [15, p. 538].

Observe that the depth of the partition subgroup $A_{i_0}^0 A_{i_1}^1 \cdots A_{i_{n-1}}^{n-1}$ is the smallest j such that $i_j < p^j$. Weir [15, Theorem 4] shows that a depth j partition subgroup is normal in P_n if and only if $i_k \leq p^j$ for all $k \geq j$.

Lemma 12.2. T_j' is the partition subgroup $A_{p^j}^{j+1} \cdots A_{p^j}^{n-1}$.

Proof. Suppose that $k \leq s \leq n-1$. Weir [15, Lemma 2] shows that if $x \in T_k \setminus T_{k+1}$ then the smallest normal subgroup of P_{s+1} containing $[A^s, x]$ is $A_{p^k}^s$. This and the fact that A^j is abelian imply the result. \square

Proposition 12.3. Suppose that $N = A_{i_j}^j A_{i_{j+1}}^{j+1} \cdots A_{i_{n-1}}^{n-1}$ is a depth j normal partition subgroup of P_n . Then the following three statements are equivalent:

- (1) N has a complement in P_n .
- (2) N has an H -invariant complement in P_n .
- (3) $i_j = 0$ and $i_k \in \{0, p^j\}$ for all $j \leq k \leq n-1$.

Proof. 1) and 2) are equivalent by Proposition 10.4, and it suffices to show that 3) is equivalent to $\bar{N} = N/T'_j$ being a direct summand of the $\mathbb{F}_p P_j$ -module \bar{T}_j , and NP'_n/P'_n not being a proper subgroup of $T_{j+1}P'_n/P'_n$.

Since $A_{p^j}^j = 1$, the $\mathbb{F}_p P_j$ -module $\bar{N} = N/T'_j$ is the direct sum

$$\bar{N} = \bigoplus_{k=j}^{n-1} A_{i_k}^k / A_{p^j}^k.$$

Now, $A_{i_k}^k / A_{p^j}^k$ is uniserial of length $p^j - i_k$, whereas \bar{T}_j is a direct sum of several copies of a length p^j uniserial module. By Lemmas 7.4 and 7.3 it follows that \bar{N} is a direct summand of \bar{T}_j if and only if $i_k \in \{0, p^j\}$ for all $j \leq k \leq n-1$. Finally, if $i_j = 0$ then NP'_n/P'_n is not subgroup of $T_{j+1}P'_n/P'_n$, let alone a proper subgroup; whereas $i_j = p^j$ would mean that N has depth $j+1$, a contradiction. \square

Lemma 12.4. *If $N \leq P_n$ has a complement then there is a partition subgroup $Q \leq P_n$ such that N and Q have a common H -invariant complement.*

Proof. From the proof of Proposition 12.3 one sees that partition subgroups with complements always fall into Case 1) of Remark 11.2, with $Z = \{\sigma_k \mid i_k = p^j\}$. Conversely, every $Z \subseteq \{\sigma_{j+1}, \dots, \sigma_{n-1}\}$ occurs in this way.

That leaves Case 2): $\langle \sigma_0, \dots, \sigma_j \rangle$ is a complement of the partition subgroup $T_{j+1} = A_0^{j+1} \cdots A_0^{n-1}$, which has depth $j+1$ rather than j . \square

APPENDIX A. ABELIAN SUBGROUPS OF LARGEST SIZE

Proposition A.1. *Let p be an arbitrary prime, $n \geq 2$ and P_n a Sylow p -subgroup of S_{p^n} . Set*

$$d = \max\{|A| \mid A \leq P_n, A \text{ abelian}\}, \quad \text{and} \\ \mathcal{M} = \{A \leq P_n \mid A \text{ abelian and } |A| = d\}.$$

Then

- (1) $d = p^{n-1}$, even in the case $n = 1$.
- (2) If p is odd then $\mathcal{M} = \{A^{n-1}\}$.
- (3) If $p = 2$ then $|\mathcal{M}| = 3^{2^{n-2}}$, and every $A \in \mathcal{M}$ lies in $T_{n-2} \cong (D_8)^{2^{n-2}}$.
- (4) (see [5, Thm 4.4.6]) If $p = 2$ then $\{C \in \mathcal{M} \mid C \leq P_n\} = \{A^{n-1}, B, W\}$, where $B \cong (C_4)^{2^{n-2}}$ is the characteristic subgroup of Proposition 1.8 and W is conjugate to A^{n-1} under the action of the outer automorphism group. Moreover, B is the only exponent four homocyclic group in \mathcal{M} .

Proof. 1): For $n = 1$ we have $d = p$, since $P_1 \cong C_p$, so assume $n \geq 2$. Then $P_n \cong P_{n-1} \wr C_p = Q \rtimes C_p$ for $Q = P_{n-1}^p$; and $d \geq p^{n-1}$ since A^{n-1} is abelian.

Now suppose $n = 2$. If $C \leq P_2$ is abelian with $C \not\leq Q$ then C contains some $x \in P_2 \setminus Q$. As $Q = A^1$ is abelian, conjugation by x acts on $Q = (C_p)^p$ by permuting the p factors cyclically. Hence $C_Q(x)$ is the diagonal subgroup of $(C_p)^p$, which is cyclic of order p . Since $C \cap Q \leq C_Q(x)$ we have

$$|C| = p |C \cap Q| \leq p^2 \leq p^p = |A^1|.$$

So if p is odd, then $|C| < |A^1|$, whence $d = p^p$ and $\mathcal{M} = \{A^1\}$. If $p = 2$, then $P_2 \cong D_8$, so $d = 2^2$ and \mathcal{M} consists of the three maximal subgroups of D_8 .

Now suppose $n > 2$. Again let $C \leq P_n$ be abelian with $C \not\leq Q$. Set $D = C \cap Q$. From $|P_n : Q| = p$ it follows that $|C : D| = p$. Now, $D \leq Q = P_{n-1}^p$, so we may consider the projection D_i onto the i th factor P_{n-1} . Then each D_i is abelian, and $D \leq \bar{D} = \prod_{i=1}^p D_i$.

Pick $x \in C \setminus D$; then x normalizes D , hence conjugation by x permutes the D_i transitively, and so $|D_i| = |D_1|$ for all i . Moreover $D \cap D_1 = 1$: for conjugation by x fixes D pointwise, but it also maps every $1 \neq y \in D_1$ into one of the other factors D_i . Hence $|\bar{D} : D| \geq |D_1|$ and so $|D| \leq |D_1|^{p-1}$.

By induction we have $|D_1| \leq p^{p^{n-2}}$. Hence

$$|C| = p|D| \leq p|D_1|^{p-1} \leq p^{p^{n-1}-p^{n-2}+1} < p^{p^{n-1}} = |A^{n-1}|.$$

2), 3): The the proof of 1) deals with the case $n = 2$, so assume $n \geq 3$. Let $C \in \mathcal{M}$. Then $C \leq Q$ by the proof of 1), so as above we have $C \leq \bar{D} = \prod_{i=1}^p D_i$, with D_i the projection of C onto the i th factor of Q . As \bar{D} is abelian we have $C = \bar{D}$ by maximality, and again by maximality, D_i lies in the \mathcal{M} for P_{n-1} . These two cases follow by induction.

4): T_{n-2} is the direct product of 2^{n-2} copies of D_8 , each of which has three order four subgroups; two being elementary abelian, and one cyclic of order four. Proposition 1.8 shows that B is characteristic. As $P_n \cong D_8 \wr P_{n-2}$ and P_{n-2} acts by permuting the factors D_8 transitively, a normal subgroup must have the same projection onto every copy D_8 : hence there are only three normal subgroup. Finally, let $\alpha \in \text{Aut}(D_8)$ be the automorphism which interchanges the two elementary abelian subgroups of rank two: α can be constructed as an inner automorphism in D_{16} . Then the automorphism $\alpha \wr \text{Id}$ of P_n interchanges the two elementary abelian normal subgroups in \mathcal{M} . \square

Corollary A.2. P_n has p -rank p^{n-1} for all primes p . \square

REFERENCES

- [1] Y. Berkovich. Some consequences of Maschke's theorem. *Algebra Colloq.*, 5(2):143–158, 1998.
- [2] Y. Berkovich. *Groups of prime power order. Vol. 1*, volume 46 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2008.
- [3] Yu. V. Bodnarchuk. Structure of the group of automorphisms of a Sylow p -subgroup of the symmetric group S_{p^n} ($p \neq 2$). *Ukrainian Math. J.*, 36(6):512–516, 1984.
- [4] H. Cárdenas and E. Lluís. El normalizador del p -grupo de Sylow del grupo simétrico S_{p^n} [The normalizer of the Sylow p -group of the symmetric group S_{p^n}]. *Bol. Soc. Mat. Mexicana (2)*, 9:1–6, 1964.
- [5] S. Covoello. *Minimal parabolic subgroups in the symmetric groups*, M.Phil. Thesis, University of Birmingham, 1998.
- [6] Yu. V. Dmítruk. Structure of Sylow two-subgroups of the symmetric group of degree 2^n . *Ukrainian Math. J.*, 30(2):117–124, 1978.
- [7] Yu. V. Dmítruk and V. I. Sushchanskii. Structure of Sylow 2-subgroups of the alternating groups and normalizers of Sylow subgroups in the symmetric and alternating groups. *Ukrainian Math. J.*, 33(3):235–241, 1981.
- [8] D. Gorenstein, *Finite groups*, Chelsea Publ. Co., New York, 1980.
- [9] L. Héthelyi, E. Horváth, *Galois actions on blocks and classes of finite groups*, *J. Algebra* **320** (2008) 660–679.
- [10] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, Heidelberg, New York 1967.

- [11] L. Kaloujnine. La structure des p -groupes de Sylow des groupes symétriques finis. *Ann. Sci. École Norm. Sup. (3)*, 65:239–276, 1948.
- [12] H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Springer Univertext, 2004.
- [13] C. R. Leedham-Green and S. McKay. *The structure of groups of prime power order*, volume 27 of *London Mathematical Society Monographs. New Series*. Oxford University Press, Oxford, 2002. Oxford Science Publications.
- [14] V. D. Mazurov. Finite groups with metacyclic Sylow 2-subgroups. *Sib. Math. J.*, 8(5):733–745, 1967.
- [15] A. J. Weir. The Sylow subgroups of the symmetric groups. *Proc. Amer. Math. Soc.* **6** (1955), 534–541.

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